Pick’s Formula and Euler’s Theorem

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Abstract:
The Pick’s formula and Euler's theorem exhibits beautiful relationships between the number of fundamental units of any figure to the area and the number of triangles contained in it. This paper presents the general overview on these formulae and proposes a modified version of Euler's theorem.

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1. Some Definitions

Here are some of the definitions used ahead in the paper:

a. **Polygons:**

A polygon is any closed figure whose area can be calculated.

b. **Edges:**

Edges are line segments which individually or in combination with other line segments form a polygon.

c. **Vertices:**

Vertices are the points at which two or more edges meet with each other.

d. **Faces:**

Faces are surfaces that are enclosed by edges on all the sides.

e. **Convex Polygons:**

When the line segment joining any two points in the inside of a polygon remain in inside of the polygon, the polygon is called a convex polygon.

f. **Concave Polygons:**

Any polygon that is not a convex polygon is called a concave polygon.

g. **Triangle:**

A figure formed by any three non-collinear points joined together with straight line segments is called triangles.

h. **Elementary Triangle:**

Any triangle that has no internal points and the vertices are the only boundary points they have. A special case of an elementary triangle is when these vertices are integral lattice points and in this case, the area of such a triangle is half.

2. Triangulation in Polygons

Theorem 1. Any n-sided polygon can be triangulated into elementary triangles.

Proof:

Base Case: *No. of sides = 2*

Let's assume two line segments AB and BC meeting at point B.

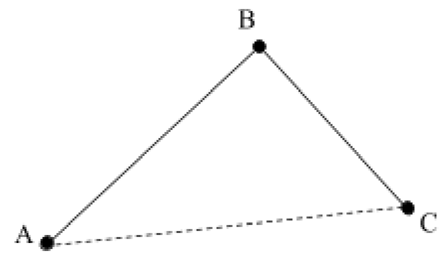


fig. 1

For this case, the vertices A and C can be connected to form an elementary triangle (from def^{n}).

∴ Base case of *no. of sides = 2* is true.

Inductive Case: Let there be a n-sided polygon *(a)* and let's assume that triangulation works for *(a)*.

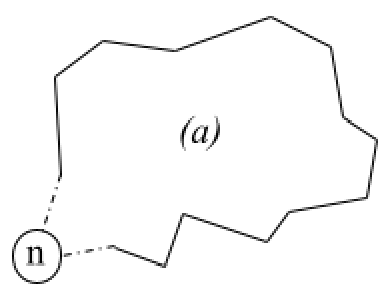


fig. 2.1

Now, let's add an extra edge AB from any vertex A in *(a)*. This results in the figure:

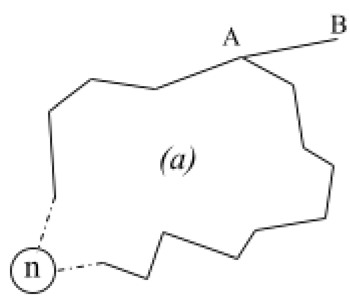


fig. 2.2

Note that there exists at least one point C such that when B and C are joined with a line segment, an elementary triangle ABC, let's call it *(b)*, is formed.

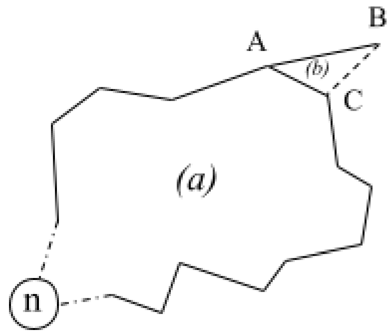


fig. 2.3

We already know that triangulation is possible in *(a)*, and from the above step, we know that triangulation is true in *(b)*. This implies that triangulation is true for the entire polygon consisting of both (*a)* and *(b)*.

∴ Inductive case is also true.

Altogether, by the principle of mathematical induction, triangulation is true for any n-sided polygon. □

2.1 Algorithm for Triangulation

Let's consider a polygon *P* for triangulation. Then here is the algorithm for the triangulation of *P*:

STEP: 1 Label all the vertices of *P*. Let's say there are *n* vertices in $V = \{1, 2, 3, \ldots, n\}$.

STEP: 2 Start at any one of the vertices, let's say 1, and find one point $x \in V$ such that the distance between the points is exactly 2 and when joined with one another using a line segment, the line segment remains completely within *P* without intersections with any other existing edge/line segment. Now, join them.

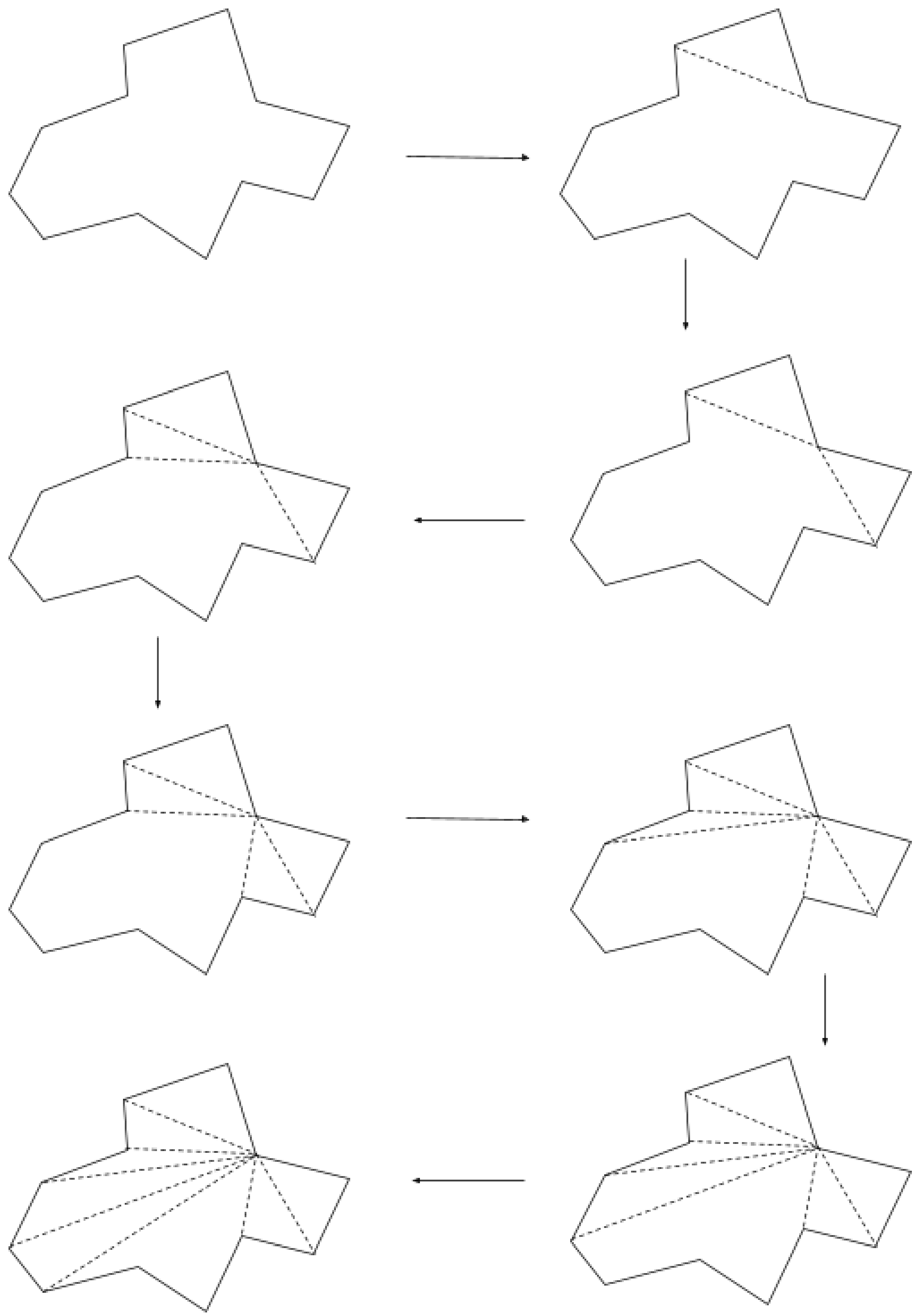
STEP: 3 Repeat STEP 2 until there exists no vertice $x \in V'$ (where V' is the set of all the vertices of *P* including the newly formed ones from STEP 2), such that the conditions mentioned in STEP 2 are satisfied.

STEP: 4 Now, move to the one of the adjacent vertices ie. vertices with a distance of one, and repeat STEP 2 and STEP 3 for this point.

STEP 5: Repeat STEP 4 until all the vertices are covered.

Def^{n}: *The distance between two vertices in a polygon is the number of edges between the two vertices.*

Here is an example showing the above algorithm:



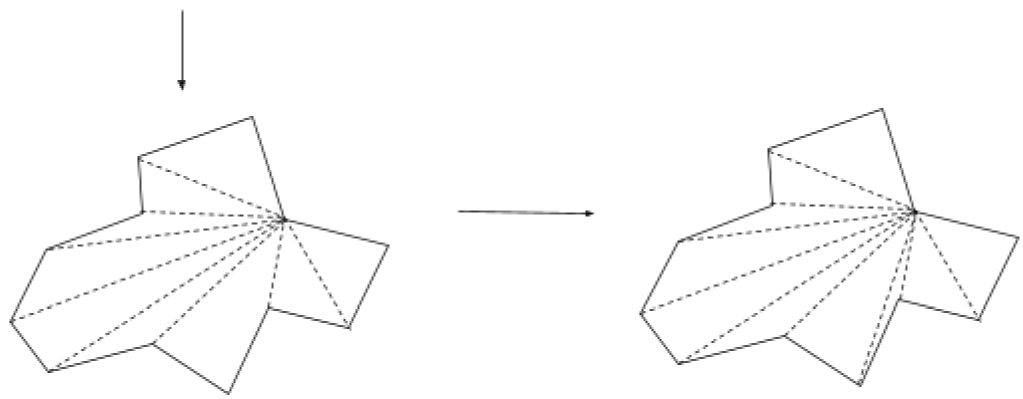


fig.2.4

3. Pick's Formula

3.1 Pick's Formula in 2-D

Theorem 2. For any n-sided polygon in 2-D with lattice points as vertices, the relationship between the number of boundary points *(B)*, number of internal points *(I)*, and area of the polygon can be given by:

$$A = I + \frac{B}{2} - 1$$

...(i)

Proof:

From *Theorem 1.*, we know any polygon can be triangulated and the least number of triangulations achievable is $T = 1$. Thus:

<u>Base Case:</u> $T = 1$

fig. 3

For $T = 1$, $I = 0$ and $B = 3$. From *eq.(i)*,

$$A = I + \frac{B}{2} - 1$$

$$A = 0 + \frac{3}{2} - 1$$

$$A = \frac{1}{2}$$

And from *fig.1*, $b = 1$ and $h = 1$, then:

$$A = \frac{1}{2} \cdot b \cdot h$$

$$A = \frac{1}{2} \cdot 1 \cdot 1$$

$$A = \frac{1}{2}$$

∴ Base case of $T = 1$, is true.

Inductive Case: Let there be two polygons P and Q with arbitrary number of boundary and internal points. Let there be n common points between the two figures.

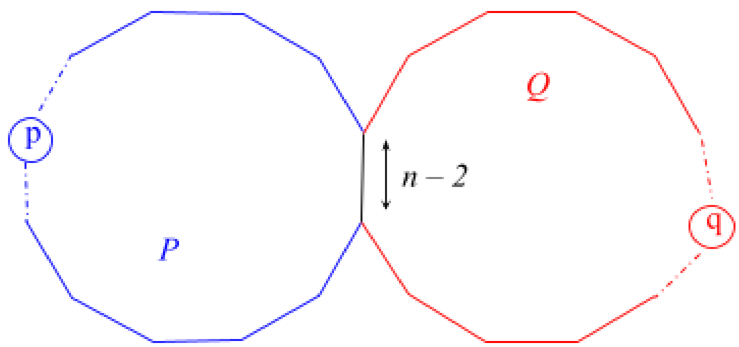


fig. 4

Assuming the formula is true for figures P and Q, we need to show that it is true for the big figure PQ. From *eq.(i)*:

$$A(P) = I_p + \frac{B_p}{2} - 1$$

$$A(Q) = I_q + \frac{B_q}{2} - 1$$

From *fig. 4*:

$$I_{pq} = I_p + I_q + (n - 2)$$

$$B_{pq} = B_p + B_q - 2(n - 2) - 2$$

then, from *eq.(i)*:

$$A(PQ) = I_p + I_q + (n - 2) + \frac{B_p + B_q - 2(n-2) - 2}{2} - 1$$

$$A(PQ) = I_p + I_q + (n - 2) - (n - 2) + \frac{B_p + B_q}{2} - 1 - 1$$

$$A(PQ) = I_p + I_q + \frac{B_p}{2} + \frac{B_q}{2} - 1 - 1$$

$$A(PQ) = I_p + \frac{B_p}{2} - 1 + I_q + \frac{B_q}{2} - 1$$

$$A(PQ) = A(P) + A(Q)$$

∴ Inductive case is also true.

Altogether, by the principle of mathematical induction, the theorem is true for all n-sided polygons. □

3.2 Exploring Pick's Formula in 3-D

Upon some exploration, it can be quickly realized that in a 3D lattice plane, to find the volume of a 3-D figure, one needs 3 variables instead of two. Let these variables be I for internal points, B for boundary points and F for face points. Then for cuboidal figures, we can write the relationship between them as:

$$V = I + \frac{B}{4} + \frac{F}{2} - 1$$

...(ii)

4. Euler's Theorem

4.1 Euler's Theorem in 2-D

Theorem 3. For any n-sided polygon in 2-D, the relationship between the number of edges *(E)*, number of vertices *(V)*, and number of triangles contained in the polygon can be given by:

$$E = V + T - 1$$

...(iii)

Proof:

From *Theorem 1.*, we know any polygon can be triangulated and the least number of triangulations achievable is *T = 1*. Thus:

Base Case: *T = 1*

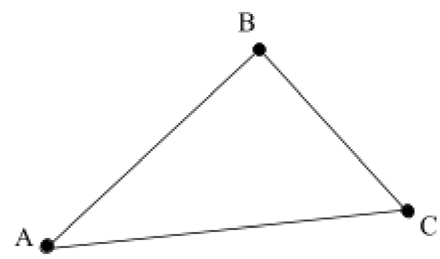


fig. 5

From *fig. 5*, by observation and definition of vertices and edges, it can be noted that *E = 3*, *V = 3* and *T = 1*. Then, from *eq.(iii)*:

$$E = V + T - 1$$
$$3 = 3 + 1 - 1$$
$$3 = 3 + 0$$
$$3 = 3$$

∴ Base case of *T = 1*, is true.

Inductive Case: *T = m* is true ⇒ *T = m + 1* is true.

Let's assume there exists a polygon having $E = e_m$, $V = v_m$ and a triangulation such that *T = m* and *eq.(iii)* is true for the polygon. This implies:

$$E = V + T - 1$$
$$e_m = v_m + m - 1$$

is true.

...(iv)

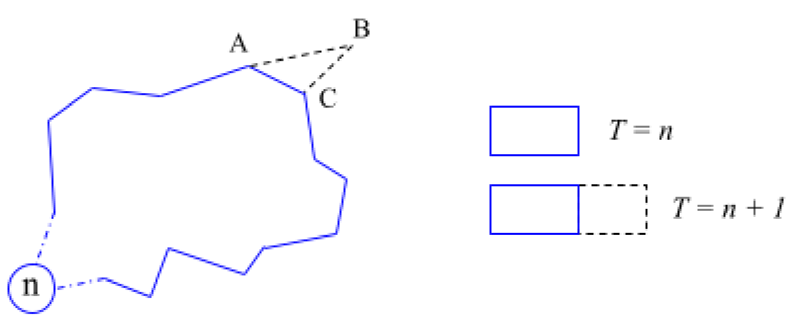


fig. 6

Upon adding one extra triangle to the polygon, there is an increase in the number of edges by two and an increase in the number of vertices by one, ie.,

$$E = e_m + 2$$

$$V = v_m + 1$$

$$T = m + 1$$

Then:

$$E = V + T - 1$$

$$e_m + 2 = (v_m + 1) + (m + 1) - 1$$

$$e_m + 2 = (v_m + 1) + m + (1 - 1)$$

$$e_m + 2 = (v_m + 1) + m + 0$$

$$e_m + 2 = (v_m + 1) + m$$

$$e_m + 2 = v_m + (1 + m)$$

$$e_m + 2 = v_m + (m + 1)$$

$$e_m + 2 = (v_m + m) + 1$$

$$e_m + 2 - 2 = (v_m + m) + 1 - 2$$

$$e_m + 0 = (v_m + m) - 1$$

$$e_m = v_m + m - 1$$

which is true according to *eq.(iv)*.

$\therefore$ $T = m$ is true $\Rightarrow$ $T = m + 1$ and the inductive case is also true.

Altogether, by the principle of mathematical induction, the theorem is true for all n-sided polygons. □

4.2 Exploring Euler's Theorem in 3-D

Translating the Euler's Formula in 2-D to 3-D, we can say that the relationship between the number of edges *(E)*, the number of vertices *(V)*, and the number of faces *(F)* can be given by:

$$E = V + F - 2$$

...(v)

5. Modified Euler's Theorem

5.1 Some Other Definitions

Here are some other definitions that are specific to this proposition:

a. Faces in 2-D:

A face in 2-D is defined as an area enclosed by edges, which is separate from other areas. Here are some examples:

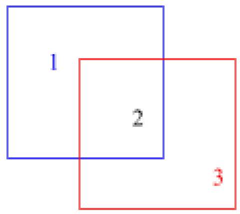


fig. 7 : F = 3

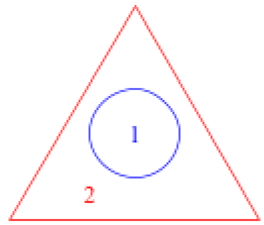


fig. 8 : F = 2

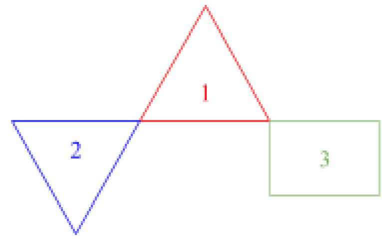


fig. 9 : F = 3

b. Stand-Alone Polygons:

If a polygon included in another polygon does not have any intersection or overlapping with the outer polygon, it is called a stand-alone polygon. Here are some examples:

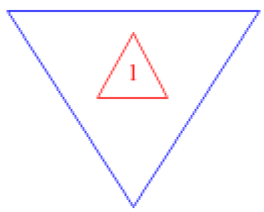


fig. 10 : S = 1

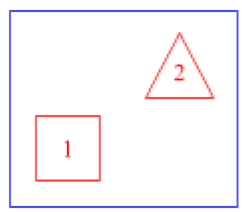


fig. 11 : S = 2

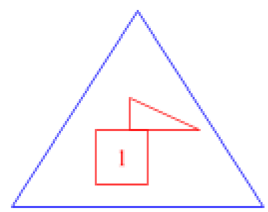


fig. 12 : S = 1

c. Edges and Vertices in a Circle:

i. Stand-Alone Circles

Stand-alone circles will have infinite edges and infinite vertices ie. E = V. To make this simpler, we can consider that each stand-alone circle has one vertex and one edge. Here is an example of a stand-alone circle:

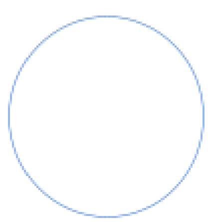


fig. 13 : V = 1, E = 1

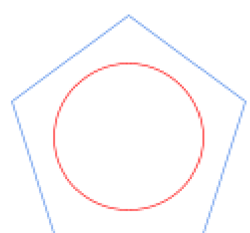


fig. 14 : V(circle) = 1, E (circle) = 1

ii. **Other Circles**

For any other circle that intersects with another polygon, the number of vertices is equal to the number of points at which the circle intersects with the polygon and number of edges is equal to the number of line-segments between the vertices. Here are some such examples:

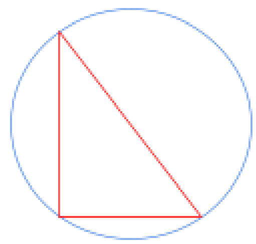


fig. 15 :
V(Circle) = 3
E(Circle) = 3

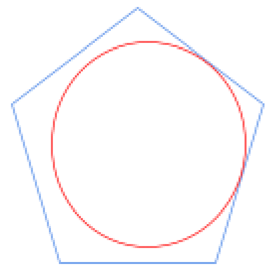


fig. 16 :
V(Circle) = 2
E(Circle) = 2

5.2 The Theorem

Theorem 4. For any combination of one or more n-sided polygons in 2-D, the relationship between the number of edges *(E)*, number of vertices *(V)*, number of faces *(F)* and the number of stand-alone polygons *(S)* contained in the polygon can be given by:

$$E = V + F - S - 1 \qquad \ldots(v)$$

Proof:
To prove *eq.(v)*, let's split the hypothesis into two cases:

<u>*Case (1):*</u> *S = 0*

Let's say there exists a polygon P_0 such that it has E_0 number of edges, V_0 number of vertices, F_0 number of faces and the number of stand-alone figures $S = 0$. Then, from *eq.(v)* we get:

$$E = V + F - S - 1$$
$$E_0 = V_0 + F_0 - 1$$

From *Theorem 1.*, we know that any polygon can be triangulated and thus, P_0 can be triangulated too. With every step in the triangulation algorithm, there is an increase in the number of edges by one and an increase in the number of faces by one.

This implies, if there are p intersections, at the end of the triangulation algorithm, there is an increase in the number of edges by p, an increase in the number of faces by p and all the faces are now translated into triangles. Therefore:

$$E_0 + p = V_0 + T_0 + p - 1$$

$$E_0 = V_0 + T_0 - 1$$

which is true for any polygon according to *Theorem 2.*

Case (2): $S \neq 0$

Let's define a polygon S_0 such that it is made up of a n-sided polygon P containing or intersecting other polygons Q_k such that none of them are stand-alone polygons.

Let the number of intersections between a polygon Q_k and P be $n(Q_k)$. Let the number of edges in S_0 be E_0, the number of vertices in S_0 be V_0, the number of faces in S_0 be F_0 and the number of stand-alone polygons be zero. Then, from the *case (1)*, we know that:

$$E_0 = V_0 + F_0 - 1$$

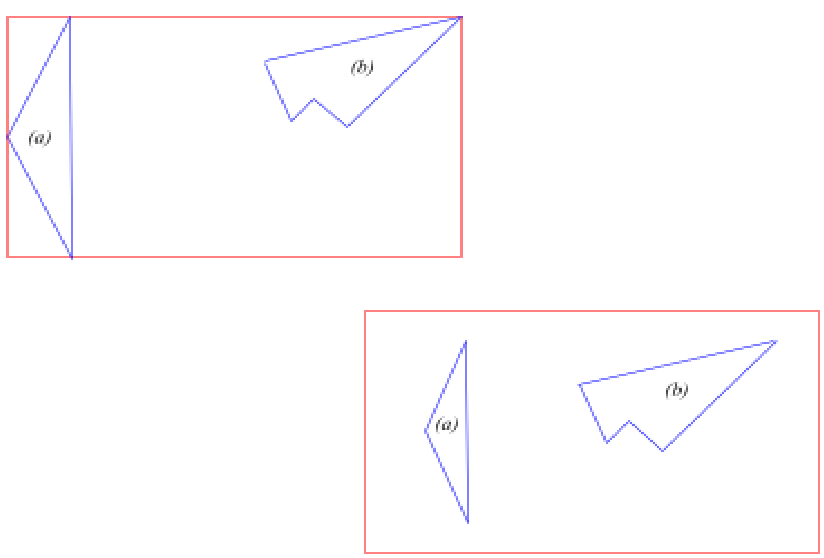


fig. 17

It can be noted that there exists only two types of intersections in between the Q_i and P, and they are:

i. Intersections at Edges (α):

When Q_k intersects with P at one of the edges of P, then it is called an intersection at an edge.

ii. Intersections at Vertices (β):

When Q_k intersects with P at one of the vertices of P, then it is called an intersection at an vertex.

Note that the total number of intersections for $k > 0$ polygons is:

$$\left(\sum_{i=1}^{k} n(Q_i)\right) = n(\alpha) + n(\beta).$$

Let's now convert all the k internal polygons Q_k into k stand-alone polygons. In order to do this, we need to disconnect the intersections that each internal polygon Q_k and this results in some changes in the number of edges and vertices in the polygon S_0.

In order to compute these changes, let's consider each type of intersection separately.

<u>Case *(2.a)*:</u> Intersections at Edges (α)

In this case, every time we disconnect an internal polygon Q_k from P, the number of vertices remains the same while the number of edges reduces by one. This is because every time we make an intersection on an edge, the edge is divided in two, resulting in an extra edge and upon removing this intersection, the number of edges goes back to the former count.

After all such intersections are disconnected, a new polygon $\overline{S_0}$ is formed such that the number of edges in it is $E_1 = E_0 - n(\alpha)$, and the number of vertices in it is not affected.

<u>Case *(2.b)*:</u> Intersections at Vertices (β)

In this case, every time we disconnect an internal polygon Q_k from P, the number of edges remains the same while the number of vertices increases by one.

This is because every time we make an intersection on a vertex, two different vertices become the same and the number of vertices reduces by one. When this intersection is disconnected, the number of vertices goes back to the former count.

After all such intersections are disconnected, a new polygon $\overline{\overline{S_0}}$ is formed such that the number of vertices in it is $V_1 = V_0 + n(\beta)$, and the number of edges is not affected.

Let's now combine these two cases. Then, the new polygon S_1 is formed such that the number of vertices $V_1 = V_0 + n(\beta)$, the number of edges in it is $E_1 = E_0 - n(\alpha)$.

Now, let's consider the changes in the number of faces in the polygon S_0 caused due to the disconnection of the interior polygons. Every time an internal polygon with p intersections is disconnected, then the number of faces is reduced by $p - 1$ and hence, when all k polygons are disconnected, the number of faces is reduced by:

$$\sum_{i=1}^{k} n(Q_i) - 1.$$

Then, the number of faces in S_1 is:

$$F_1 = F_0 - \left(\sum_{i=1}^{k} n(Q_i) - 1\right)$$

$$F_1 = F_0 - \left(n(Q_1) - 1\right) - \left(n(Q_2) - 1\right) - ... - \left(n(Q_k) - 1\right)$$

$$F_1 = F_0 - \left(\sum_{i=1}^{k} n(Q_i)\right) + k$$

$$F_1 = F_0 - (n(\alpha) + n(\beta)) + k$$

From the above deductions:

$$V_1 = V_0 + n(\beta)$$

$$E_1 = E_0 - n(\alpha)$$

$$F_1 = F_0 - (n(\alpha) + n(\beta)) + k$$

$$E_0 = V_0 + F_0 - 1$$

$$E_0 - n(\alpha) = V_0 + F_0 - 1 - n(\alpha) + n(\beta) - n(\beta) + k - k$$

$$E_0 - n(\alpha) = V_0 + n(\beta) + F_0 - n(\alpha) - n(\beta) + k - k - 1$$

$$E_0 - n(\alpha) = V_0 + n(\beta) + F_0 - (n(\alpha) + n(\beta)) + k - 1 - k$$
$$E_1 = V_1 + F_1 - 1 - k$$

where E_1 is the number of edges, V_1 is the number of vertices, F_1 is the number of faces, and k is the number of stand-alone polygons contained within S_1. Therefore, the proposition is true for all $k > 0$. $\square$

Citations:

The above paper was written without making any references to existing material about the topics as an exercise to think deeper about simple mathematical concepts. However, we recognize and appreciate the existence of similar papers and theories:

Davis. (2003, October 27). Pick's Theorem. Retrieved 2023, from http://www.geometer.org/mathcircles/pick.pdf

Suri. (n.d.). Triangulation. Retrieved 2023, from https://ima.udg.edu/~sellares/ComGeo/TriangulationSuri.pdf

Champanerkar. (n.d.). Euler's Polyhedral Formula. Retrieved 2023, from https://www.math.csi.cuny.edu/abhijit/623/euler_slides.pdf